MATH 579 Exam 5 Solutions

1. For all integers k, n with $1 \leq k \leq n$, prove that $\left\{ {n \atop k} \right\} = \left\{ {n-1 \atop k-1} \right\} + k\left\{ {n-1 \atop k} \right\}$.

   We count partitions of $[n]$ into k blocks. One way is directly, which is the LHS. Another way is to focus attention on element n. If it is in a block by itself, there are $\left\{ {n-1 \atop k-1} \right\}$ ways to put the remaining elements into $k-1$ blocks. If it is not, the remaining elements form k blocks, which can be done in $\left\{ {n-1 \atop k} \right\}$ ways, and then one of the k blocks must be chosen for n.

2. Prove that $S(n, 3) = \frac{3^{n-1}-2^n+1}{2}$ for all $n \in \mathbb{N}$. Hint: induction on n.

   For $n = 1, 2$ we have both sides equal to 0. Otherwise, we use $\left\{ {n \atop 3} \right\} = \left\{ {n-1 \atop 2} \right\} + 3\left\{ {n-1 \atop 3} \right\}$. We recall that $\left\{ {n \atop 2} \right\} = 2^{n-1}-1$, so $\left\{ {n-1 \atop 2} \right\} = 2^{n-2}-1$. Also, by the inductive hypothesis, $\left\{ {n-1 \atop 3} \right\} = \frac{3^{n-2}-2^{n-1}+1}{2}$. Combining, $\left\{ {n \atop 3} \right\} = 2^{n-2} - 1 + 3 \cdot \frac{3^{n-2}-2^{n-1}+1}{2} = \frac{2^{n-1}-2+3^{n-1}-3\cdot 2^{n-1}+3}{2} = \frac{3^{n-1}-2\cdot 2^{n-1}+1}{2}$, as desired.

3. (5-10 points) Use difference calculus to calculate $\sum_{k=1}^{19} k^3 - k$.

   We have $k^3 = \left\{ {3 \atop 1} \right\}(k)_1 + \left\{ {3 \atop 2} \right\}(k)_2 + \left\{ {3 \atop 3} \right\}(k)_3 = (k)_1 + 3(k)_2 + (k)_3$, while $k^1 = \left\{ {1 \atop 1} \right\}(k)_1 = (k)_1$. Hence $k^3 - k = 3(k)_2 + (k)_3$. Our sum is $\sum_1^{20} 3(k)_2 + (k)_3 \delta k = (k)_3 + (1/4)(k)_4|_1^{20} = (20)_3 + (1/4)(20)_4 - ((1)_3 + (1/4)(1)_4) = 20 \cdot 19 \cdot 18 + (1/4)20 \cdot 19 \cdot 18 \cdot 17 = 35910$.

4. (5-10 points) We call a number "happy" if it yields remainder 2 upon division by 3. (Happy numbers include 2,5, and 8). How many compositions are there of 100 into 5 happy parts?

   We have $100 = x_1 + x_2 + x_3 + x_4 + x_5$, a composition into happy parts. We write $x_i = 2 + 3y_i$, for some nonnegative integers y_i, hence we have $90 = 3y_1 + 3y_2 + 3y_3 + 3y_4 + 3y_5$, which is equivalent to $30 = y_1 + y_2 + y_3 + y_4 + y_5$, a weak composition of 30 into 5 parts. There are $\left(\!\binom{5}{30}\!\right) = \binom{34}{30} = 46376$ of these.

5. (5-12 points) How many compositions are there of 30 into 5 parts, one of which is happy and four of which are odd?

   Note first that four odd parts have even sum, and 30 is even, hence the happy part must be even. Conveniently, there is no confusion about odd vs. happy, since no part can be both. We now assume that the happy part is first, and in the end we will multiply by 5 to account for the position of the happy part. We may write the happy part as $2 + 3k$, but $k = 2j$ must be even since the happy part is even, so in fact the happy part is $2 + 6j$. Hence we have $30 = (2 + 6j) + x_1 + x_2 + x_3 + x_4$. Hence we need a composition of $(28 - 6j)$ into four odd parts. We write $x_i = 1 + 2y_i$, so we have $24 - 6j = 2y_1 + 2y_2 + 2y_3 + 2y_4$, equivalently $12 - 3j = y_1 + y_2 + y_3 + y_4$. Hence we seek weak compositions of $12 - 3j$ into 4 parts, of which there are $\left(\!\binom{4}{12-3j}\!\right) = \binom{15-3j}{12-3j}$. We need this for all nonnegative j, so $\binom{15}{12} + \binom{12}{9} + \binom{9}{6} + \binom{6}{3} + \binom{3}{0} = 780$, hence to account for the 5 positions of the happy part, there are 3900 desired compositions.